

A NOTE ON L^p -NORMS OF QUASI-MODES

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Dedicated to Shanzhen Lu on the occasion of his seventy fifth birthday.

ABSTRACT. In this note we show how improved L^p -estimates for certain types of quasi-modes are naturally equivalent to improved operator norms of spectral projection operators associated to shrinking spectral intervals of the appropriate scale. Using this, one can see that recent estimates that were stated for eigenfunctions also hold for the appropriate types of quasi-modes.

1. Introduction and main results

Let (M, g) be a compact boundaryless Riemannian manifold of dimension $n \geq 2$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the frequencies associated with Δ and let $e_j(x)$ be an orthonormal basis of eigenfunctions,

$$(1.1) \quad (\Delta^2 + \lambda_j^2)e_j(x) = 0, \quad \text{and} \quad \int_M |e_j|^2 dV = 1.$$

Here $dV = dV_g$ and $\Delta = \Delta_g$ denote the volume element and Laplace-Beltrami operator associated with (M, g) , respectively. Also define the ‘unit-band’ spectral projection operators (i.e. for frequency intervals of unit length) by

$$(1.2) \quad \chi_{[\lambda, \lambda+1]} f(x) = \sum_{\lambda_j \in [\lambda, \lambda+1]} E_j f,$$

where E_j denotes the projection onto the j -th eigenspace, i.e.,

$$E_j f(x) = \langle f, e_j \rangle e_j(x),$$

with $\langle \cdot, \cdot \rangle$ being the $L^2(dV)$ -inner product.

Many recent works are concerned with L^p norms of eigenfunctions and their relations to the geometry of the geodesics of (M, g) . The first author proved universal bounds on L^p norms of eigenfunctions [12]. For large p the bounds are saturated by zonal (rotationally invariant) spherical harmonics on the standard sphere S^n , which peak at the poles, and for low p they are saturated by elliptic Gaussian beams which concentrate on closed geodesics (see [11]). In general, a Riemannian manifold fails to have either kind of eigenfunction and that suggests that the L^p norms of eigenfunctions are strictly smaller on a typical (M, g) than in the case of the sphere. The theme of our works [17], [16], [20] is that for $p = \infty$ (or for ‘large p ’), the L^p bounds can only be saturated if there exist (partial) poles or self-focal points. Less is known about necessary geometric conditions for existence of eigenfunctions saturating the low L^p norms. It is plausible that that existence of a stable elliptic geodesic is necessary for saturation of low L^p norms. We refer to [14], [18], [19] for some recent results and to [15], [22], [23] for surveys of the problem.

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Many of the techniques and results concerning L^p norms of eigenfunctions (‘modes’) apply equally well to quasi-modes or approximate eigenfunctions. Quasi-modes are linear combinations of eigenfunctions with eigenvalues concentrated in a short interval. Specifically, if λ ranges over a sequence $\{\mu_k\}$ tending to $+\infty$, then we say that $\{\psi_\lambda\}$ (with $\lambda = \mu_k$) is a quasi-mode of order s if

$$(1.3) \quad \int_M |\psi_\lambda|^2 dV = 1, \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} = O(\lambda^{-s}).$$

Note then that, by our convention of using λ^2 instead of λ in the second part of (1.7), the quasi-modes ψ_λ are associated with frequencies λ . Roughly speaking, the higher the s , the shorter the band of frequencies required to construct the quasi-mode. We would like to give bounds on L^p norms of quasi-modes of a given order s and relate them to the geometry of geodesics. For instance, we may ask if there exist quasi-modes of order s which saturate the L^p bounds and if they resemble zonal spherical harmonics for high p or Gaussian beams for low p . Of course, quasi-modes peaking at a point or concentrating on a closed geodesic exist much more often than actual eigenfunctions with these properties. But as in the case of modes, there are necessary geometric conditions on (M, g) for the construction of quasi-modes of order s which saturate the L^p bounds of [12]. It is not hard to see that there are no conditions if one lets $s = -1$, since the Schwartz kernel $\chi_{[\lambda, \lambda+1]}(\cdot, x)$ itself saturates the high L^p bounds and the spectral projection $\chi_{[\lambda, \lambda+1]}(\delta_\gamma)$ of the delta-function on a closed geodesic saturates the low L^p bounds. The question is whether one can find necessary geometric conditions under which quasi-modes with spectrum in just slightly shrinking intervals can saturate the L^p bounds. If so, Riemannian manifolds (M, g) which fail to satisfy the conditions are said to have “improved estimates” or to satisfy “sub-convexity bounds”.

As this discussion indicates, closely related to estimates of modes and quasi-modes are mapping norm estimates of spectral projections for intervals. If

$$(1.4) \quad \sigma(p, n) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}, \end{cases}$$

then it was shown in [12] that

$$(1.5) \quad \|\chi_{[\lambda, \lambda+1]} f\|_{L^p(M)} \leq C \lambda^{\sigma(p, n)} \|f\|_{L^2(M)}, \quad \lambda \geq 1, \quad p \geq 2.$$

These unit-band spectral projection estimates turn out to be sharp and cannot be improved on *any* compact boundaryless Riemannian manifold, as was shown in [13].

In this note we consider spectral projections for shrinking intervals. As with quasi-modes, the question is to find geometric conditions on (M, g) under which one can improve or ‘break convexity’ for the mapping norm estimate. Breaking convexity bounds on mapping norms of spectral projections means finding a rate $\varepsilon(\lambda)$ of shrinking intervals $[\lambda, \lambda + \varepsilon(\lambda)]$ and a condition on (M, g) for which one can prove

$$(1.6) \quad \|\chi_{[\lambda, \lambda + \varepsilon(\lambda)]} f\|_{L^p(M)} \leq C \sqrt{\varepsilon(\lambda)} \lambda^{\sigma(p, n)} \|f\|_{L^2(M)},$$

assuming that $\varepsilon(\lambda) > 0$ and $\varepsilon(\lambda)$ tends monotonically to zero as $\lambda \rightarrow \infty$. Here, as before, $\sigma(p, n)$ is as in (1.4). The dependence of (1.6) on $\varepsilon(\lambda)$ is sharp in view of the fact that (1.5) cannot be improved. To see this one uses orthogonality and the dual version of (1.5).

In this note, we do not give new geometric conditions. Rather our purpose is to show that the conditions for improving mapping norm estimates or sub-convexity bounds for spectral projection operators associated to shrinking intervals are equivalent to the conditions improved bounds for all quasi-modes of the related order. The arguments are implicit in [5] and [16] but have not previously been made explicit. We shall focus on two different types of quasi-modes that naturally occur, although the methods will handle other types as well.

1.1. Quasi-modes of order $o(\lambda)$. The first type we call quasi-modes of order $o(\lambda)$ or “fat quasi-modes”. In place of (1.7) they satisfy

$$(1.7) \quad \int_M |\psi_\lambda|^2 dV = 1, \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} = o(\lambda).$$

These occur at least implicitly in several recent works, including [2], [3], [4],[5],[6], [14], [16], [17] and [20]. We would like to give conditions on (M, g) which break the convexity bounds, i.e. prohibit fat quasi-modes from saturating the L^p bounds of [12]. Our first result states that breaking convexity bounds for fat quasi-modes is equivalent to breaking convexity bounds for mapping norms of spectral projections for shrinking intervals. That is, we replace the unit intervals $[\lambda, \lambda + 1]$ by shrinking intervals $[\lambda, \lambda + o(1)]$ and seek sufficient conditions on (M, g) so that if $\sigma(p, n)$ is as in (1.4), then

$$(1.8) \quad \|\chi_{[\lambda, \lambda + o(1)]}\|_{L^2(M) \rightarrow L^p(M)} = o(\lambda^{\sigma(p, n)}).$$

By this we of course mean that for every $\varepsilon > 0$ there is a $0 < \delta < 1$ and a $\Lambda < \infty$ (both depending on ε) so that

$$(1.9) \quad \|\chi_{[\lambda, \lambda + \delta]}\|_{L^2(M) \rightarrow L^p(M)} \leq \varepsilon \lambda^{\sigma(p, n)}, \quad \lambda > \Lambda,$$

where

$$\chi_{[\lambda, \lambda + \delta]} f = \sum_{\lambda_j \in [\lambda, \lambda + \delta]} E_j f,$$

and

$$\|\chi_{[\lambda, \lambda + \delta]}\|_{L^2(M) \rightarrow L^p(M)}$$

denotes the $L^2(M) \rightarrow L^p(M)$ operator norm of $\chi_{[\lambda, \lambda + \delta]}$. Improved bounds (1.8) were shown to be generically true for large exponents p in [17] and [16], and in many cases in [2] and [19] for $2 < p < \frac{2(n+1)}{n-1}$,

The condition (1.7) says that in an L^2 -sense the spectrum of the ψ_λ are concentrated in intervals of length $o(1)$ about $\lambda = \mu_k$ as $\mu_k \rightarrow \infty$. In view of this, the following result naturally links $o(\lambda^{\sigma(p, n)})$ estimates for these $o(\lambda)$ quasi-modes with the improved L^p -bounds in (1.9).

Theorem 1.1. *Suppose that $n \leq 3$ and $p > 2$, $n = 4$ and $2 < p < \infty$ or $n \geq 5$ and $2 < p \leq \frac{2n}{n-4}$. Then (1.8) is valid if and only if whenever λ ranges over a sequence $\mu_k \rightarrow \infty$ we have*

$$(1.10) \quad \|\psi_\lambda\|_{L^p(M)} = o(\lambda^{\sigma(p, n)}),$$

assuming that the ψ_λ satisfy the $o(\lambda)$ quasi-mode condition (1.7). If $n = 4$ and $p = \infty$ or $n \geq 5$ and $\frac{2n}{n-4} < p \leq \infty$, the same conclusion is valid if we replace (1.7)

by the condition that

$$(1.11) \quad \int_M |\psi_\lambda|^2 dV = 1,$$

$$\text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} + \lambda^{2-n(\frac{1}{2}-\frac{1}{p})} \|\chi_{[2\lambda, \infty)}\psi_\lambda\|_{L^p(M)} = o(\lambda),$$

where

$$\chi_{[2\lambda, \infty)} f = \sum_{\lambda_j \geq 2\lambda} E_j f$$

denotes the projection of f onto frequencies $\lambda_j \in [2\lambda, \infty)$.

Note that for all exponents $p > 2$ if $n \leq 3$ or for $2 < p < \infty$ when $n = 4$ or $2 < p \leq \frac{2n}{n-4}$ the second term in the last condition in (1.11) is dominated by the first term by Sobolev embeddings. Thus, for these exponents, the condition (1.11) is equivalent to the more succinct version (1.7). On the other hand for the other exceptional exponents $p > 2$ if $n \geq 4$, this term is needed for the conclusion as was argued in [16]. This is just due to Sobolev considerations.

1.2. Quasi-modes of order zero. The second type of quasi-mode we consider are those of order zero, which satisfy

$$(1.12) \quad \int_M |\psi_\lambda|^2 dV = 1, \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} = O(1),$$

assuming, as above, that λ ranges over a sequence $\mu_k \rightarrow \infty$. We refer to [24] for background and references.

Note that the second condition in (1.12) says that, in an L^2 -sense, the spectrum of the ψ_λ are concentrated in intervals of length $1/\lambda$ about λ . Thus, the following result is the natural analog of Theorem 1.1.

Theorem 1.2. *Suppose that $n \leq 3$ and $p > 2$, $n = 4$ and $2 < p < \infty$ or $n \geq 5$ and $2 < p \leq \frac{2n}{n-4}$. Then*

$$(1.13) \quad \|\chi_{[\lambda, \lambda+1/\lambda]}\|_{L^2(M) \rightarrow L^p(M)} = o(\lambda^{\sigma(p,n)}),$$

where $\sigma(p, n)$ is as in (1.4) if and only if whenever λ ranges over a sequence $\mu_k \rightarrow \infty$ we have

$$(1.14) \quad \|\psi_\lambda\|_{L^p(M)} = o(\lambda^{\sigma(p,n)}),$$

assuming that the ψ_λ satisfy the zero order quasi-mode condition (1.13). If $n = 4$ and $p = \infty$ or $n \geq 5$ and $\frac{2n}{n-4} < p \leq \infty$, the same conclusion is valid if we replace (1.13) by the condition that

$$(1.15) \quad \int_M |\psi_\lambda|^2 dV = 1,$$

$$\text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} + \lambda^{2-n(\frac{1}{2}-\frac{1}{p})} \|\chi_{[2\lambda, \infty)}\psi_\lambda\|_{L^p(M)} = O(1),$$

where $\chi_{[2\lambda, \infty)}$ is as above.

1.3. A unifying principle. As we shall see both of these Theorems are a consequence of the following result, which is a variation of the ones in [5].

Proposition 1.3. *Suppose that $n \leq 3$ and $p > 2$, $n = 4$ and $2 < p < \infty$ or $n \geq 5$ and $2 < p \leq \frac{2n}{n-4}$. Then there is a uniform constant $C = C(M, g)$ depending only on our n -dimensional Riemannian manifold (M, g) so that for $0 < \delta < 1$ and sufficiently large λ we have*

$$(1.16) \quad \|f\|_{L^p(M)} \leq C \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times (\|f\|_{L^2(M)} + (\delta\lambda)^{-1} \|(\Delta + \lambda^2)f\|_{L^2(M)}).$$

Similarly,

$$(1.17) \quad \|f\|_{L^p(M)} - \|\chi_{[2\lambda, \infty)}f\|_{L^p(M)} \leq C \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times (\|f\|_{L^2(M)} + (\delta\lambda)^{-1} \|(\Delta + \lambda^2)f\|_{L^2(M)}),$$

for the remaining exponents $p > 2$, i.e., $p = \infty$ if $n = 4$ and $\frac{2n}{n-4} < p \leq \infty$ if $n \geq 5$.

2. Proof that Proposition 1.3 implies Theorems 1.1 and 1.2

We shall start by going over the simple argument which shows how Proposition 1.3 implies the two theorems.

Let us first suppose that (1.8) is valid. Thus, given $\varepsilon > 0$ we can find a fixed $\delta = \delta(\varepsilon) \in (0, 1)$ so that (1.9) is valid. In other words, we have that

$$\sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \leq \varepsilon \lambda^{\sigma(p, n)},$$

if $\lambda \gg 1$ is large enough. By (1.16), we then have that

$$(2.1) \quad \|f\|_{L^p(M)} \leq C \varepsilon \lambda^{\sigma(p, n)} (\|f\|_{L^2(M)} + (\delta\lambda)^{-1} \|(\Delta + \lambda^2)f\|_{L^2(M)}).$$

Since we have fixed $\delta \in (0, 1)$, if we take $f = \psi_\lambda$, where the ψ_λ satisfy (1.7), we conclude that (2.1) yields

$$(2.2) \quad \|\psi_\lambda\|_{L^p(M)} \leq 2C \varepsilon \lambda^{\sigma(p, n)},$$

for large enough λ . Thus, since $\varepsilon > 0$ is arbitrary, we obtain (1.10) assuming that (1.8) is valid and $n \leq 3$ or $2 < p < \infty$ if $n = 4$ or $2 < p \leq \frac{2n}{n-4}$ if $n \geq 5$. Since

$$(2.3) \quad n\left(\frac{1}{2} - \frac{1}{p}\right) - 1 < \sigma(p, n)$$

for all exponents $p > 2$, we similarly obtain (1.10) for the remaining exponents from (1.17) assuming that (1.8) is valid and that the ψ_λ satisfy (1.11).

To prove the remaining half of Theorem 1.1, let

$$(2.4) \quad V_{[\lambda, \lambda+\delta]} = \{f \in L^2(M) : E_j f = 0, \lambda_j \notin [\lambda, \lambda + \delta]\}$$

denote the span of the eigenfunctions whose frequencies lie in $[\lambda, \lambda + \delta]$. It then follows that

$$(2.5) \quad \|\chi_{[\lambda, \lambda+\delta]}\|_{L^2(M) \rightarrow L^p(M)} = \sup_{f \in V_{[\lambda, \lambda+\delta]}, \|f\|_2=1} \|f\|_{L^p(M)},$$

and

$$(2.6) \quad \|(\Delta + \lambda^2)f\|_{L^2(M)} \leq 3\delta\lambda, \quad \text{if } f \in V_{[\lambda, \lambda+\delta]} \text{ and } \|f\|_{L^2(M)} = 1,$$

assuming that $0 < \delta < 1$ and $\lambda \geq 1$. Note also that for such δ, λ , we have $\chi_{[2\lambda, \infty)} f = 0$ if $f \in V_{[\lambda, \lambda+\delta]}$. Thus if (1.10) is valid assuming either (1.7) or (1.11) as λ ranges over any sequence $\mu_k \rightarrow \infty$, we conclude that (1.8) must hold, completing the proof of Theorem 1.1.

The proof of Theorem 1.2 is very similar. First, if (1.13) is valid then by (2.1) with $\delta = 1/\lambda$, we conclude using (1.16) again that we must have (2.2) if the ψ_λ satisfy (1.12) and $n \leq 3$ and $p > 2$, $2 < p < \infty$ if $n = 4$ or $2 < p \leq \frac{2n}{n-4}$ if $n \geq 5$. We similarly obtain (1.14) assuming (1.15) from (1.17) for the remaining exponents $p > 2$, since for these we have

$$n(\frac{1}{2} - \frac{1}{p}) - 2 < \sigma(p, n).$$

Since the converse part of Theorem 1.2 follows from the corresponding proof of this part of Theorem 1.1 (using $\delta = 1/\lambda$ in (2.5) and (2.6)), we obtain the second theorem as well.

3. Proof of Proposition 1.3

We now prove Proposition 1.3. We first note that by the Sobolev estimate $\|u\|_p \leq C\|u\|_{W^{k,2}}$ (with $k = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$) and orthogonality, if $2 < p < \infty$ and $n \leq 4$ or $2 < p \leq \frac{2n}{n-4}$ and $n \geq 5$, we have

$$\begin{aligned} (3.1) \quad \|\chi_{[2\lambda, \infty)} f\|_{L^p(M)} &\leq C\|(-\Delta)^{\frac{n}{2}(\frac{1}{2} - \frac{1}{p})} \chi_{[2\lambda, \infty)} f\|_{L^2(M)} \\ &\leq C\lambda^{n(\frac{1}{2} - \frac{1}{p}) - 2} \|(\Delta + \lambda^2) \chi_{[2\lambda, \infty)} f\|_{L^2(M)} \\ &\leq C\lambda^{n(\frac{1}{2} - \frac{1}{p}) - 2} \|(\Delta + \lambda^2) f\|_{L^2(M)}. \end{aligned}$$

Similarly, for every $\varepsilon > 0$ we have

$$(3.2) \quad \|\chi_{[2\lambda, \infty)} f\|_{L^\infty(M)} \leq C_\varepsilon \lambda^{\frac{n}{2} - 2 + \varepsilon} \|(\Delta + \lambda^2) f\|_{L^2(M)}, \quad \text{if } n \leq 3.$$

Also, as we noted before, (1.5) cannot be improved, and, therefore, there must be a uniform constant $c > 0$, which is independent of $0 < \delta < 1$, so that if λ is large enough we have

$$c\delta\lambda^{\sigma(p,n)} \leq \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)},$$

and so

$$\begin{aligned} (3.3) \quad \lambda^{\sigma(p,n)-1} \|(\Delta + \lambda^2) f\|_{L^2(M)} \\ \leq C \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times (\delta\lambda)^{-1} \|(\Delta + \lambda^2) f\|_{L^2(M)}. \end{aligned}$$

Therefore, if we recall (2.3) and combine (3.1)-(3.3), we conclude that if $2 < p \leq \infty$ and $n \leq 3$ or $2 < p < \infty$ and $n = 3$ or $2 < p \leq \frac{2n}{n-4}$ and $n \geq 5$, then there is a uniform constant C , which is independent of $0 < \delta < 1$, so that for sufficiently large λ we have

$$\|\chi_{[2\lambda, \infty)} f\|_{L^p(M)} \leq C(\delta\lambda)^{-1} \|(\Delta + \lambda^2) f\|_{L^2(M)} \times \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)}.$$

From this, we deduce that we would have (1.16) and (1.17) if we could show that

$$(3.4) \quad \begin{aligned} & \|\chi_{[0,2\lambda]} f\|_{L^p(M)} \\ & \leq C \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times (\|f\|_{L^2(M)} + (\delta\lambda)^{-1} \|(\Delta + \lambda^2)f\|_{L^2(M)}), \end{aligned}$$

for all exponents $2 < p \leq \infty$ if $n \geq 2$ and δ and λ are as above.

To prove this we write

$$(3.5) \quad \chi_{[0,2\lambda]} f = \chi_{[\lambda-\delta, \lambda+\delta]} f + \sum_{k=1}^{\infty} \sum_{|\lambda-\lambda_j| \in (\delta k, \delta(k+1)]} \chi_{(0,2\lambda)} E_j f.$$

Clearly

$$(3.6) \quad \begin{aligned} \|\chi_{[\lambda-\delta, \lambda+\delta]} f\|_{L^p(M)} & \leq \|\chi_{[\lambda-\delta, \lambda+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \|f\|_{L^2(M)} \\ & \leq 2 \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times \|f\|_{L^2(M)}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \left\| \sum_{|\lambda-\lambda_j| \in (\delta k, \delta(k+1)]} \chi_{(0,2\lambda)} E_j f \right\|_{L^p(M)} \\ & \leq \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times \left\| \sum_{|\lambda-\lambda_j| \in (\delta k, \delta(k+1)]} \chi_{(0,2\lambda)} E_j f \right\|_{L^2(M)} \\ & \leq (\delta k \lambda)^{-1} \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times \left\| \sum_{|\lambda-\lambda_j| \in (\delta k, \delta(k+1)]} (\Delta + \lambda^2) E_j f \right\|_{L^2(M)}. \end{aligned}$$

Therefore, by orthogonality and an application of the Cauchy-Schwarz inequality, we deduce that the $L^p(M)$ -norm of the last term in (3.5) is

$$\leq C \sup_{\mu \in [\lambda/2, 2\lambda]} \|\chi_{[\mu, \mu+\delta]}\|_{L^2(M) \rightarrow L^p(M)} \times (\delta\lambda)^{-1} \|(\Delta + \lambda^2)f\|_{L^2(M)}.$$

From this and (3.6), we obtain (3.4), which completes the proof of Proposition 1.3.

4. Applications to breaking convexity bounds

In this section, we survey some recent results on breaking convexity bounds on mapping norms for $\chi_{[\lambda, \lambda+\varepsilon(\lambda)]}$ of the form (1.6). By Proposition 1.3, breaking such convexity bounds is equivalent to proving estimates

$$(4.1) \quad \|\psi_\lambda\|_{L^p(M)} \leq C \sqrt{\varepsilon(\lambda)} \lambda^{\sigma(p,n)},$$

for quasi-modes satisfying

$$(4.2) \quad \|\psi_\lambda\|_{L^2(M)} \leq 1 \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} \leq \varepsilon(\lambda)\lambda,$$

assuming that $2 < p \leq \infty$ if $n \leq 3$, $2 < p < \infty$ if $n = 4$ and $2 < p \leq \frac{2n}{n-3}$ if $n \geq 5$. For the remaining exponents $p > 2$ if $n = 4$ or $n \geq 5$, we can obtain (4.1) if we also assume that

$$(4.3) \quad \lambda^{2-n(\frac{1}{2}-\frac{1}{p})} \|\chi_{[2\lambda, \infty)} \psi_\lambda\|_{L^p(M)} = O(1).$$

Note that the second condition in (4.2) is natural for (1.6) since it says that, in an L^2 -sense, most of the spectrum of the ψ_λ are concentrated in intervals of size $O(\varepsilon(\lambda)\lambda)$ about λ .

4.1. Nondissipative focal points exist if convexity cannot be broken. It is proved in [17, Lemma 6.1] and [16, Theorem 2] that for generic (M, g) there exists a monotonically decreasing function $\varepsilon(\lambda) \rightarrow 0$ (the rate depending on (M, g)) so that (1.6) is valid. Thus, in this case, we also get (4.1) for $p = \infty$ and the above type of quasi-modes. However, $\varepsilon(\lambda)$ is not described explicitly.

If one *cannot* break the convexity bound on mapping norms, then one has

$$(4.4) \quad \|\chi_{[\lambda+o(1)]}\|_{L^2(M) \rightarrow L^\infty(M)} = \Omega(\lambda^{\frac{n-1}{2}}),$$

by which we mean that there is a uniform constant $c > 0$ so that

$$(4.5) \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-\frac{n-1}{2}} \|\chi_{[\lambda, \lambda+\delta]}\|_{L^2(M) \rightarrow L^\infty(M)} \geq c,$$

whenever $\delta > 0$. In the real analytic case in [20] we found a necessary and sufficient condition for this. Specifically, we showed that (4.4) is valid if and only if our real analytic manifold (M, g) has a nondissipative self-focal point p . ‘Self-focal’ means that there exists a time $T(p)$ so that, for all $\xi \in S_p^*M$ (the unit tangent space), the geodesic $\gamma_{p,\xi}(t)$ with initial data (p, ξ) returns to p at time $T(p)$. The geodesic is not assumed to smoothly close up (i.e. to be a closed geodesic), but just to loop back. The map $\Phi_p : S_p^*M \rightarrow S_p^*M$ taking the initial direction ξ to the terminal direction $\gamma'_{x,\xi}(T(p))$ is known as the first return map. It induces a unitary operator $U_p : L^2(S_p^*M) \rightarrow L^2(S_p^*M)$ defined by

$$U_p(f) = f(\Phi_p(\xi)) \sqrt{J_p(\xi)},$$

where $\Phi_p^*|d\xi| = J_p(\xi)|d\xi|$ is the Jacobian with respect to the induced Riemannian volume density on S_p^*M . We say that a self-focal point is dissipative if U_p has no L^2 eigenfunction. Otherwise it is nondissipative.

To obtain (4.4) we showed that if there exists a nondissipative self-focal point x_0 , then if $0 < \delta < 1$ we have

$$(4.6) \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-(n-1)} \sum_{j=0}^{\infty} \rho(\delta^{-1}(\lambda - \lambda_j)) |e_j(x_0)|^2 \geq c_0,$$

for some $c_0 > 0$. Here, $\rho \in \mathcal{S}(\mathbb{R})$ is any test function satisfying

$$\rho, \hat{\rho} \geq 0, \quad \text{supp } \hat{\rho} \subset [-1, 1], \quad \rho(s) \geq 1, \quad \text{if } |s| \leq 1.$$

We prove (4.6) by using the von Neumann ergodic theorem for U_p together with arguments adapted from [7], [9] and [10]. We obtain (4.5) from (4.6) due to the fact that

$$\|\chi_{[\lambda, \lambda+\delta]}\|_{L^2(M) \rightarrow L^\infty(M)}^2 = \sup_{x \in M} \sum_{\lambda_j \in [\lambda, \lambda+\delta]} |e_j(x)|^2.$$

Conversely, if (4.5) is valid one of course gets that, for every $0 < \delta < 1$,

$$\limsup_{\lambda \rightarrow \infty} \left(\sup_{x \in M} \lambda^{-(n-1)} \sum_{j=0}^{\infty} \rho(\delta^{-1}(\lambda - \lambda_j)) |e_j(x)|^2 \right) \geq c_0 > 0.$$

It follows from (4.4) and Proposition 1.3 that if our real analytic (M, g) has a nondissipative self-focal point then there is a $o(\lambda)$ sequence of quasi-modes satisfying

$$(4.7) \quad \|\psi_\lambda\|_{L^\infty(M)} = \Omega(\lambda^{\frac{n-1}{2}}).$$

We proved this in [20] by using a special case of Proposition 1.3.

4.2. Existence of a quasi-mode of order zero with maximal growth of L^∞ norms? A natural question in [20] is whether (4.4) implies existence of a sequence of quasi-modes of order zero which saturate the L^∞ bounds. That is, if one has an L^2 eigenfunction of U_p at a self-focal point p , can one construct from it a quasi-mode of order zero with maximal sup-norm growth? This would definitely be the case if the L^2 eigenfunction were C^∞ . In that case, we could use it as the symbol for a quasi-mode of order zero. In [16] such a construction was made in the special case of a self-focal point for which $\Phi_p = Id$, i.e. for which all of the loops at p were smoothly closed.

In view of Theorem 1.2, existence of a zeroth order quasi-mode of maximal sup norm growth would imply that

$$(4.8) \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-\frac{n-1}{2}} \|\chi_{[\lambda, \lambda+1/\lambda]}\|_{L^2(M) \rightarrow L^\infty(M)} > 0.$$

4.3. Logarithmic shrinking of intervals for non-positively curved manifolds. In the case of non-positively curved manifolds we can be more specific about the shrinking rate of intervals and of L^p estimates.

Implicit in [1] is the fact that (1.6) is valid with $\varepsilon(\lambda) = 1/\ln(\lambda)$ if (M, g) has nonpositive sectional curvatures (see [15, §3.6]). This result was extended to all exponents $\frac{2(n+1)}{n-1} < p \leq \infty$ in [8] (see also [5, §4]). Thus, for such manifolds we have

$$(4.9) \quad \|\psi_\lambda\|_{L^p(M)} \leq C\lambda^{\sigma(p,n)}/\sqrt{\ln \lambda}, \quad \frac{2(n+1)}{n-1} < p \leq \infty,$$

assuming that

$$\|\psi_\lambda\|_{L^2(M)} \leq 1 \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(M)} \leq C\lambda/\ln \lambda,$$

as well as (4.3) if $n = 4$ or $n \geq 5$ and p is large.

The logarithmic estimate (4.9) reflects the hyperbolicity of the geodesic flow on negatively curved manifolds. It is obtained more generally in microlocal analysis of quasimodes around hyperbolic closed geodesics. It is plausible that one does not need global curvature assumptions but just the assumption that all closed geodesics of (M, g) are hyperbolic. In the case of *quantum integrable* Laplacians (such as a surface of revolution shaped like a peanut), it is possible to construct eigenfunctions which concentrate microlocally on a hyperbolic closed geodesic and whose L^p norms in tubes around the closed geodesic saturate the inequality (4.9). We refer to [23, §5.1] for further discussion and references.

It is possible to construct quasi-modes ψ_λ on negatively curved surfaces which partially concentrate on hyperbolic closed geodesics. A quasi-mode satisfying (4.2) with $\varepsilon(\lambda) = 1/\ln(\lambda)$ has been constructed recently by S. Brooks [4] on a compact hyperbolic surface whose microlocal lift (or defect measure) has mass $\geq \delta$ for a certain $\delta > 0$ along a closed geodesic γ . The estimate of (4.9) gives an upper bound on its L^p norms, and for $p = 6 + \varepsilon$ is precisely the order of magnitude of a quasi-mode associated to a hyperbolic geodesic. It would be interesting to see if Brooks' quasi-mode saturates the bound (4.9).

4.4. $\varepsilon(\lambda) = \lambda^{-\varepsilon(p,n)}$ on a flat torus. In the case of the standard n -torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, it was also shown in [5] that we have estimates of the form

$$\|\chi_{[\lambda, \lambda+\lambda^{-\varepsilon(p,n)}]}\|_{L^2(\mathbb{T}^n) \rightarrow L^p(\mathbb{T}^n)} \leq \lambda^{\sigma(p,n)-\varepsilon(p,n)/2}, \quad \frac{2(n+1)}{n-1} < p \leq \infty,$$

for certain powers $\varepsilon(p, n) > 0$. Therefore, by the above, we have

$$\|\psi_\lambda\|_{L^p(\mathbb{T}^n)} \leq C\lambda^{\sigma(p,n)-\varepsilon(p,n)/2}, \quad \frac{2(n+1)}{n-1} < p \leq \infty,$$

assuming that

$$\|\psi_\lambda\|_{L^2(\mathbb{T}^n)} \leq 1 \quad \text{and} \quad \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(\mathbb{T}^n)} \leq C\lambda^{1-\varepsilon(p,n)},$$

as well as (4.3) if p is large enough and $n \geq 4$.

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